Synchronization in the Kuramoto model: A dynamical gradient network approach

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We propose a dynamical gradient network approach to consider the synchronization in the Kuramoto model. Our scheme to adaptively adjust couplings is based on the dynamical gradient networks, where the number of links in each time interval is the same as the number of oscillators, but the links in different time intervals are also different. The gradient network in the $(n+1)$th time interval is decided by the oscillator dynamics in the nth time interval. According to the gradient network in the $(n+1)$th time interval, only one inlink's coupling for each oscillator is adjusted by a small incremental coupling. During the transition to synchronization, the intensities for all oscillators are identical. Direct numerical simulations fully verify that the synchronization in the Kuramoto model is realized effectively, even if there exist delayed couplings and external noise.

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In the past decades, the synchronization in complex networks has been a research topic in many fields [1–14]. Among many models that have been proposed to address synchronization phenomena, one of the most successful models is the Kuramoto model (KM) [3,4]. This model can be used to understand the emergence of synchronization in networks of oscillators. In particular, this model presents a second-order phase transition from incoherence to synchronization. For the synchronization on the KM, many works assumed that the couplings are constant [6–9]. Recently, some works introduced the time-varying couplings in the KM. Maistrenko *et al.* introduced the mechanism of plasticity in the KM to study the multistability, and assumed that the couplings are varied in accordance with the spike timing-dependent plasticity [14]. Ren and Zhao also proposed one time-varying coupling scheme by introducing continuously adaptive couplings, and this can enhance the synchronization in the KM. In their scheme, the couplings grow stronger for the pairs which have larger phase incoherence [5].

In this Brief Report we also consider the synchronization in the KM by introducing a dynamical gradient network (GN) approach. This study is motivated in part by the work [15,16], where the concept of GNs is introduced. GNs are directed subnetworks of an undirected "substrate" network in which each oscillator has an associated scalar potential and one outlink that points to the oscillator with the smallest (or largest) potential in the reunion of itself and its neighbors on the substrate network. The existence of gradients has been shown to play an important role in biological transport processes, such as cell migration [17]: chemotaxis, haptotaxis, and galvanotaxis. Naturally, the same mechanism will generate flows in complex networks as well [16]. In addition, GNs have been already utilized to enhance synchronization in complex networks [12]. A general weighted asymmetrical network is regarded as a superposition of a weighted symmetrical network and a weighted GN. Depending on degrees of oscillators, a weighted coupling scheme is proposed to enhance the synchronizability in networks. However, the proposed GN is static, which means its structure is time independent.

Differing from the static GN in Ref. [12], the GNs developed in this Brief Report are dynamical, which means that the GNs in different time intervals are different. Here the KM consists of a population of N coupled oscillators where the phase $\theta_i(t)$ of the ith oscillator evolves in time according to

$$\frac{d\theta_i}{dt} = w_i + \sum_j \Lambda_{ij} A_{ij} \sin(\theta_j - \theta_i), \quad i = 1,2, \ldots ,N, \tag{1}$$

where w_i are natural frequencies distributed with a given probability density $g(w)$, and A_{ij} is the binary adjacency matrix representing the topology of networks. Further, $\Lambda_{ij}>0$ is the coupling strength of the inlink (i,j) pointing from oscillator j to oscillator i if they are connected. Denote Γ_i as the index set of neighbors of oscillator i.

The KM (1) can be solved in terms of the order parameter $r(t)$ that measures the extent of synchronization as

$$r(t)e^{l\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{l\theta_j(t)}, \tag{2}$$

where $l^2=-1$, $\Psi(t)$ stands for an average phase, and the parameter $0 \le r(t) \le 1$. Obviously, if $r(t)=1$, the synchronization in the KM is realized. The parameter $r(t)$ given by Eq. (2) has been widely used [3–5,7].

We first segment the time interval $[t_0, \infty)$ into

$$[t_0, \infty) = \bigcup_{n\ge 1} [t_{n-1}, t_n), \tag{3}$$

where $t_n=t_0+nT$, t_0 is the transient time, and the length T of intervals is chosen suitably. For the parameter $r(t)$, we define one local order parameter for oscillator i in the interval $[t_{n-1}, t_n)$ as follows:

$$r^{i,n} = \frac{1}{T}\int_{t_{n-1}}^{t_n} r_i(t)dt, \tag{4}$$

with $r_i(t)e^{l\Psi_i(t)} = \frac{1}{k_i+1}\sum_{j\in\Gamma_i\cup\{i\}} e^{l\theta_j(t)}$, where k_i is the degree of oscillator i. The parameter $r^{i,n}$ can measure the local synchronization extent among oscillator i and its neighbors. If $r^{i,n_0}=1$ for certain n_0, oscillator i and its neighbors are locally synchronized in the interval $[t_{n_0-1}, t_{n_0})$. For the network of oscillators, the extent of synchronization in our scheme is to choose the order parameter $r_0(n)$ as follows:

$$r_0(n) := \frac{1}{T}\int_{t_{n-1}}^{t_n} r(t)dt. \quad (5)$$

If there is a n_0 such that $r(n_0)=1$, we conclude that the synchronization in networks is realized.

Here we introduce an adaptive coupling scheme into the KM. Our idea to adjust the coupling Λ_{ij} in the interval $[t_n, t_{n+1})$ is based on the concept of GNs [15,16]. To define a GN at the instant $t=t_n$, we consider a network denoted by $\Sigma=(V,E_n)$, where V stands for the set of oscillators, and E_n denotes the set of links at the instant $t=t_n$. Consider a field denoted by $h^n=\{h_1^n, \ldots, h_N^n\}$ at the instant $t=t_n$, where h_i^n is the scalar assigned to oscillator i. We define the gradient $\nabla_{h_i^n}$ of the field h_i^n in oscillator i to be the directed link $\nabla_{h_i^n} =(i,\mu_i^n)$, where $\mu_i^n \in \Gamma_i$ represents one neighbor of oscillator i. At the instant $t=t_n$, the network $\Sigma_g=(V,\nabla_n)$, where ∇_n is the set of the gradients $\nabla_{h_i^n}$, is called a GN. Note that at different time instants the GNs are also different. In this Brief Report, this kind of GN is called the dynamical GN. In the GN Σ_g, the directed link (i,μ_i^n) points from oscillator μ_i^n, at which the scalar field has the minimum (or maximum) value in oscillator $\mu_i^n \in \Gamma_i$, i.e., [16],

$$\mu_i^n = \arg\min_{j\in\Gamma_i} h_j^n \quad (6)$$

to oscillator i. If several neighbors have the same scalar field, we choose only one randomly.

For the parameter (4), we choose the scalar field h_i^n as

$$h_i^n = r^{i,n}. \quad (7)$$

In the GN composed of the gradients $\nabla_{h_i^n}$, we adjust the coupling $\Lambda_{i\mu_i^n}$ of the inlink (i,μ_i^n) pointing from oscillator μ_i^n to oscillator i. Denote the coupling Λ_{ij} in the interval $[t_{n-1},t_n)$ as Λ_{ij}^n.

The coupling scheme. In the interval $[t_n,t_{n+1})$, we adaptively adjust the coupling $\Lambda_{i\mu_i^n}$ of the inlink (i,μ_i^n) in the GN $\Sigma_g=(V,\nabla_n)$ by

$$\Lambda_{i\mu_i^n}^{n+1} := \Lambda_{i\mu_i^n}^{n} + \varepsilon, \quad (8)$$

where $\varepsilon > 0$ is arbitrarily small incremental coupling. When the link (i,j) does not belong to the GN Σ_g, its coupling satisfies

$$\Lambda_{ij}^{n+1} := \Lambda_{ij}^{n}. \quad (9)$$

Now we analyze the feasibility of the above coupling scheme by the linearized dynamics of the KM. When the Kuramoto dynamics is close to the attractor, the phase differences are small, and then the sine coupling function can be approximated linear. Therefore, in the interval $[t_n,t_{n+1})$, the linearized dynamics of oscillator i can be written in the form

$$\frac{d\theta_i}{dt} = \sum_j \Lambda_{ij}^n A_{ij}(\theta_j - \theta_i) + \varepsilon(\theta_{\mu_i^n} - \theta_i). \quad (10)$$

In the above equation the last term $\varepsilon(\theta_{\mu_i^n}-\theta_i)$ is equivalent to the term $-\varepsilon(\theta_i-\theta_{\mu_i^n})$, which can be regarded as the negative feedback term for the unidirectional synchronization from oscillator μ_i^n to oscillator i. This could make the phase difference between oscillator i and its neighbor μ_i^n be smaller, which may make the synchronization in the KM be realized.

The adaptive scheme (8) and (9) can be easily extended to the KM with delayed couplings and external noise. One case is the KM described by [3,6]

$$\frac{d\theta_i}{dt} = w_i + \sum_j \Lambda_{ij}A_{ij}\sin(\theta_j - \theta_i) + \xi_i(t), \quad i=1,2,\ldots,N, \quad (11)$$

where $\xi_i(t)$ is the white noise duo to the complicated environment, with expectation and variance $\langle\xi_i(t)\rangle=0$, $\langle\xi_i(t)\xi_j(t')\rangle=2\delta_{ij}\delta(t-t')$. Another case is the KM given by [3]

$$\frac{d\theta_i}{dt} = w_i + \sum_j \Lambda_{ij}A_{ij}\sin(\theta_{j,\tau} - \theta_i) + \xi_i(t), \quad i=1,2,\ldots,N, \quad (12)$$

where the term $\theta_{j,\tau}$ represents the delayed phase $\theta_j(t-\tau)$, and τ is a constant time delay.

Our simulations are based on scale-free (SF) and small-world (SW) networks. The SF networks are generated by the Barabási-Albert model [18], where the initial network is a fully connected network with M oscillators, labeled by $i=1,\ldots,M$. At every time step a new oscillator is introduced to be connected to M existing oscillators. The probability that a new oscillator is connected to oscillator i depends on degree k_i of oscillator i, namely, $\Pi_i=k_i/\Sigma_j k_j$. After repeating for $N-M$ times, a SF network has a degree distribution $P(k)\sim k^{-3}$ and the minimal degree $k_{\min}=M$. The SW networks are generated by the Newman-Watts model [19], where the initial network is a nearest-neighbor coupled network consisting of N oscillators arranged in a ring, with oscillator i being adjacent to its neighbor oscillators, $j=1,\ldots,K/2$, and with K being even. Then one adds with probability p a connection between a pair of oscillators.

In our simulations, the initial weights for all links are zero, the natural frequencies of oscillators are uniformly distributed in the interval $[-1,1]$, the transient time is $t_0=100$ s, the length of intervals is $T=1$, and the incremental coupling is $\varepsilon=0.01$. The solution of networks is resolved using the Euler method and the time step $h=0.02$ s, and our ending condition for our scheme is $|r(n_0)-1|<10^{-2}$ for certain n_0.

We first simulate the SF networks with $N=1000$ and the SW networks with $N=1000$ and $p=0.03$ in the absence of noise. We plot the local order parameter r_0 as a function of the adjustment time n [Fig. 1(a)], and the global order parameter r as a function of the time step m $(=n/h)$ [Fig. 1(b)]. Obviously, due to our coupling scheme (8) and (9), the KM (1) can be in a synchronized state after several hundreds of the adjustment steps. In every time interval, only one inlink's

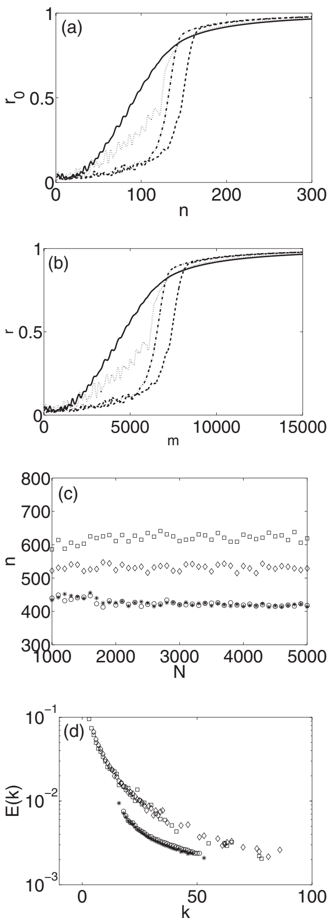

FIG. 1. Simulation results in the KM (1) without noise. The parameter $r_0(n)$ as the function of n (a), and the parameter $r(t)$ as the function of m (b), in the SF networks (solid line, $M=3$; dotted line, $M=5$) and the SW networks (dash-dotted line, $K=2$, $p=0.03$; dashed line, $K=4$, $p=0.03$). The adjustment step n as a function of the size N in networks (c), and standard deviation $E(k)$ as the function of degree k in the SF and SW networks with $N=1000$ (square, $M=3$; diamond, $M=5$; star, $K=2$, $p=0.03$; circle, $K=4$, $p=0.03$). All estimates are the result of averaging over 50 realizations.

coupling for each oscillator is adjusted by a same small incremental coupling, and the other inlinks' couplings remain constant. Hence the intensities S defined by $S=\sum_j \Lambda_{ij} A_{ij}$ for all oscillators are identical during the transition to synchronization. From Figs. 1(a) and 1(b), the extent of synchronization in the KM increases with the increasing of the intensity S given by $S=n\varepsilon$. In our coupling scheme, the intensity S is a good indicator of the synchronization in the KM. At about $n=300$, namely, $S=3$, the KM (1) is practically in a synchronized state. However, the equal intensities cannot be ensured by other adaptive coupling schemes [5,10]. The intensities in Ref. [10] strongly depend on heterogeneous degrees in the SF networks. The larger the degree of an oscillator is, the larger its intensity is.

We also discuss the synchronization in the SF and SW networks with different size. Under the same ending condition, we observe that the adjustment steps needed to synchronize the SF networks with the same M are almost identical [Fig. 1(c)]. It further means that the time $(n_0 T)$ needed to

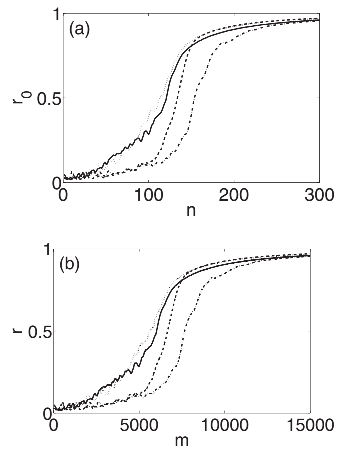

FIG. 2. Simulation results in the KM (1) with noise. The parameter $r_0(n)$ as the function of n (a), and the parameter $r(t)$ as the function of m (b), in the SF networks (solid line, $M=4$; dotted line, $M=6$) and the SW networks (dash-dotted line, $K=6$, $p=0.01$; dashed line, $K=8$, $p=0.01$). All estimates are the result of averaging over 50 realizations.

synchronize the SF networks with the same M is almost equal. We can also obtain similar results in the SW networks with the same K and p. The steps in the SW networks with the same p and a small K are almost identical while the steps in the SF networks with different M are also different. This can be in part explained by the average degree $\langle k \rangle \approx 2M$ in the SF networks and $\langle k \rangle \approx K+(N-1)p/2$ in the SW networks. When the average degree is smaller, it requires a longer time to synchronize networks.

After the ending of our scheme (8) and (9), we can also analyze the relationship between the normalized coupling matrix $G=(G_{ij})$ with $G_{ij}=\frac{\Lambda_{ij}^{n_0}}{n_0\varepsilon}A_{ij}$ and the coupling matrix $G_0=(\Lambda'_{ij}A_{ij})$ with $\Lambda'_{ij}=1/k_i$. We compute the average error $E(k)=\frac{1}{\gamma_k}\sum_{q=1}^{\gamma_k}E_q$ between G and G_0, where γ_k is the number of oscillators with the same degree k_i, and $E_q=\sqrt{\sum_{j\neq i}(G_{ij}-1/k_i)^2/k_i}$. We show that G_{ij} is almost identical to the value $1/k_i$ (or $G_{ij}\sim k_i^{-1}$) [Fig. 1(d)]. After the ending of our scheme, the couplings $\Lambda_{ij}^{n_0}$ for the inlinks of oscillator i are approximately $n_0\varepsilon/k_i$. Therefore, for the SF networks with the same M and the SW networks with the same K and p, the maximal coupling relies on the minimal degree in networks. The larger the degree of oscillator i is, the smaller the coupling $\Lambda_{ij}^{n_0}$ is.

Even if there exists noise in the KM (1), we can also obtain similar results in the SF networks with different M and the SW networks with different K and p [Fig. 2]. For the KM (12) with delayed couplings, simulation results are plotted in Fig. 3 ($\tau=1$) and Fig. 4 ($\tau=3$). Here we only plot figures on the parameters r_0 and r due to the space limitation. From these figures, the synchronization can be realized effectively. Note that there are two parameters T and ε in our scheme. Due to the weak coupling on the synchronization in the KM, the value ε in our scheme cannot be large, and the length T of intervals can be arbitrarily large. In our simulations the value ε can be chosen from the interval $[0.0001, 0.02]$. For different values of T and ε, we can obtain similar results.

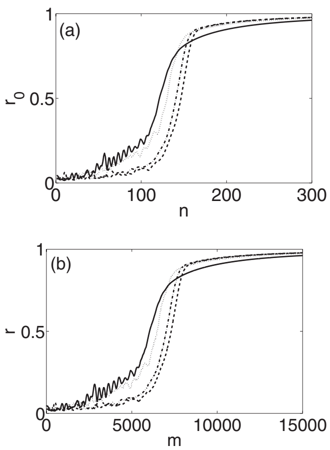

FIG. 3. Simulation results in the KM ([12]) without noise. The parameter $r_0(n)$ as the function of n (a), and the parameter $r(t)$ as the function of m (b), in the SF network (solid line, $M=4$; dotted line, $M=7$) and the SW network (dash-dotted line, $K=6$, $p=0.02$; dashed line, $K=8$, $p=0.02$). All estimates are the result of averaging over 50 realizations.

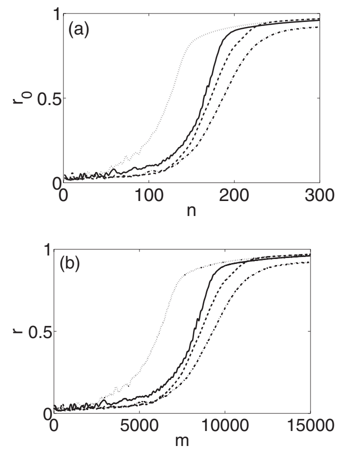

FIG. 4. Simulation results in the KM ([12]) with noise. The parameter $r_0(n)$ as the function of n (a), and the parameter $r(t)$ as the function of m (b), in the SF network (solid line, $M=4$; dotted line, $M=6$) and the SW network (dash-dotted line, $K=6$, $p=0.01$; dashed line, $K=8$, $p=0.01$). All estimates are the result of averaging over 50 realizations.

Gómez-Gardeñes *et al.* proposed another order parameter $r_{\rm link}$ to measure the extent of synchronization [8], where $r_{\rm link}=\frac{1}{2N_{\rm link}}\sum_i\sum_{j\in\Gamma_i}|\lim_{\Delta_t\to\infty}\frac{1}{\Delta_t}\int_{t_r}^{t_r+\Delta_t}e^{I[\theta_i(t)-\theta_j(t)]}dt|$, where $N_{\rm link}$ is the number of links, and t_r is a large time. The averaging time Δ_t is taken large enough to obtain good measures of the degree of coherence between each pair of physically connected oscillators. Eqs. ([4]), ([5]), and ([7]) in our scheme can be replaced by $r_{\rm link}^{i,n}=\frac{1}{k_i}\sum_{j\in\Gamma_i}|\frac{1}{T}\int_{t_{n-1}}^{t_n}e^{I[\theta_i(t)-\theta_j(t)]}dt|$, $r'_{\rm link}(n)=\frac{1}{2N_{\rm link}}\sum_i\sum_{j\in\Gamma_i}|\frac{1}{T}\int_{t_{n-1}}^{t_n}e^{I[\theta_i(t)-\theta_j(t)]}dt|$, and $h_i^n=r_{\rm link}^{i,n}$, respectively. Since numerical results are very similar to those with respect to the parameters $r^{i,n}$ and $r_0(n)$ (Figs. [1–4]), we omit figures for the space limitation.

In this Brief Report, we introduce an adaptive coupling scheme in the KM based on the dynamical GN approach. Our scheme can be applied to different variants of the KM. Compared with the adaptive couplings [5,10], our scheme has the following merits: (1) In each time interval only one inlink's coupling for each oscillator is adjusted by the same small value, which ensures that the number of inlinks to be adjusted in each time interval is only N, rather than all inlinks [5,10]. (2) In the $(n+1)$th time interval, all oscillators have the same intensities $S=n\varepsilon$. However, the adaptive couplings in Refs. [5,10] cannot ensure the equal intensities. (3) For oscillator i, the inlink to be adjusted in the $(n+1)$th time interval is decided by some scalar fields, which depend on the oscillator dynamics in the nth time interval.

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[1] R. Albert and A. L. Barabasi, Rev. Mod. Phys. **74**, 47 (2002); S. Boccaletti *et al.*, Phys. Rep. **424**, 175 (2006).
[2] A. Pikovsky *et al.*, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001); S. Boccaletti *et al.*, Phys. Rep. **366**, 1 (2002).
[3] J. Acebron *et al.*, Rev. Mod. Phys. **77**, 137 (2005); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
[4] S. H. Strogatz, Physica D **143**, 1 (2000).
[5] Q. Ren and J. Zhao, Phys. Rev. E **76**, 016207 (2007).
[6] J. Ren and H. Yang, e-print arXiv:cond-mat/0703232v2.
[7] Y. Moreno and A. F. Pacheco, Europhys. Lett. **68**, 603 (2004); A. Arenas, A. Diaz-Guilera, and C. J. Perez-Vicente, Phys. Rev. Lett. **96**, 114102 (2006).
[8] J. Gomez-Gardenes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **98**, 034101 (2007).
[9] M. Timme, Phys. Rev. Lett. **98**, 224101 (2007).
[10] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[11] C. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006); D. U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, *ibid.* **94**, 138701 (2005).
[12] X. Wang, Y. Lai, and C. Lai, Phys. Rev. E **75**, 056205 (2007).
[13] F. Radicchi and H. Meyer-Ortmanns, Phys. Rev. E **74**, 026203 (2006).
[14] Y. L. Maistrenko, B. Lysyansky, C. Hauptmann, O. Burylko, and P. A. Tass, Phys. Rev. E **75**, 066207 (2007).
[15] K. Park, Y. C. Lai, L. Zhao, and N. Ye, Phys. Rev. E **71**, 065105(R) (2005).
[16] Z. Toroczkai and K. E. Bassler, Nature (London) **428**, 716 (2004); Z. Toroczkai *et al.*, e-print arXiv:cond-mat/0408262.
[17] R. Nuccitelli, in *Pattern Formation: A Primer in Developmental Biology*, edited by G. M. Malacinski (MacMillan, New York, 1984), p. 23.
[18] A.-L. Barabasi and R. Albert, Science **286**, 509 (1999).
[19] M. E. J. Newman and D. J. Watts, Phys. Lett. A **263**, 341 (1999).